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Lesson 27: Recursive Challenge Problem—The Double and Add 5 Game

Student Outcomes

* Students learn the meaning and notation of recursive sequences in a modeling setting.
* Students use recursive sequences to model and answer problems.
* Students create equations and inequalities to solve a modeling problem.
* Students represent constraints by equations and inequalities and interpret solutions as viable or non-viable options in a modeling context.

Lesson Notes

The *double and add 5* game is *loosely* related to the Collatz conjecture—an *unsolved* conjecture in mathematics named after Lothar Collatz, who first proposed the problem in 1937. The conjecture includes a recurrence relation, *triple and add 1,* as part of the problem statement. A worthwhile activity for you and your class is to read about the conjecture online.

About the lessons: Students begin by playing the *Double and Add 5* game in a simple situation. Given a number, double it and add 5. The result of round two is the double of the result of round one, plus 5, and so on. The goal of the game is to find the smallest starting whole number, that produces a number 100 or greater in three rounds or less (Answer: ). Students are then exposed to the more difficult challenge of finding the smallest starting whole number that produces a number 1000 or greater in three rounds or less. To solve this problem, the notation of recursive sequences and recursive relations are explained, and students formalize the problem in terms of an equation, solve, interpret their answer, and validate (Answer: ).

Classwork

This challenging two-day modeling lesson (see page 61 of CCLS) about recursive sequences runs through the *problem, formulate, compute, interpret, validate, report* modeling cycle. This modeling activity involves playing a game and describing the mathematical process in the game using a recurrence relation in order to solve a more difficult version of the game. This part two lesson picks up were the last lesson left off—in this lesson students formulate, compute, interpret, validate, and report on their answers to the *Double and Add 5* game problem stated in the previous lesson.

Recall the statement of the problem from the last lesson for your students:

*Given a starting number, double it and add 5 to get the result of round one. Double the result of round one and add 5, and so on. The goal of the game is to find the smallest starting whole number that produces a result of 1000 or greater in three rounds or less.*

Example 1 (10 minutes)

The repeat of this example from the previous lesson speaks to the value and importance of students doing this work. This time require students to work individually to complete the task. Visit students as needed and ask questions that lead students to the correct formulas.

Example 1

Review Exercise 4 from the previous lesson: Using a generic initial value, , and the recurrence relation, , for , find a formula for in terms of .

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Exercise 1 (15 minutes)

(Formulation step of the modeling cycle). Ask students: Using one of the four formulas from Example 1, write an inequality that, if solved for , will lead to finding the smallest starting whole number for the *Double and Add 5* game that produces a result of 1000 or greater in three rounds or less.

**MP.2**

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**MP.4**

Exercise 1

Using one of the four formulas from Example 1, write an inequality that, if solved for , will lead to finding the smallest starting whole number for the “double and add 5” game that produces a result of 1000 or greater in 3 rounds or less.

This exercise is loaded with phrases that students will need to interpret correctly in order to formulate an equation (*do not* expect this to be easy for them). Start with simple questions and build up:

* What does mean in terms of rounds?
  + *The result of round two.*
* Write what the statement, “produce a result of 1000 or greater in two rounds,” means using a term of the sequence.
  + *The result of round two, must be greater than or equal to 1000. Ask students to write the equation, , for that statement.*
* After replacing in the inequality, with the expression in terms of , what does the numbers that satisfies the inequality, mean?
  + *The numbers that satisfies the inequality are the starting numbers for the Double and Add 5 game that produce a result of 1000 or greater in two rounds or less. The “or less” in the previous sentence is important and can be understood by thinking about the question, “Do we need two rounds to reach 1000 starting with number 999? 800? 500?”*

Let students solve forin , and let them find the smallest whole numberfor exactly two rounds *[Answer: 247].* Then continue with your questioning:

* What inequality in terms of would you write down to find the smallest starting number for the *Double and Add 5* game that produces a result of 1000 or greater in three rounds or less?
  + *.*

Exercise 2 (10 minutes)

(Compute, interpret, validate steps of the modeling cycle) . Tell students:

Exercise 2

Solve the inequality derived in Exercise 2. Interpret your answer, and validate that it is the solution to the problem, that is, show that the whole number you found results in 1000 or greater in three rounds, but the previous whole number takes four rounds to reach 1000.

**MP.2**

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**MP.3**

Students should write or say something like: I interpret or as the set of all starting numbers that reach 1000 or greater in three rounds or less. Therefore the smallest starting whole number is 121. To validate, I checked that starting with 121 results in 1003 after three rounds, whereas 120 results in 995 after three rounds.

Exercise 3 (5 minutes)

(This exercise cycles through the modeling cycle again.) Ask students:

Exercise 3

Find the smallest starting whole number for the *Double and Add 5* game that produces a result of 1,000,000 or greater in four rounds or less.

Students should write or say something like: I interpreted or as the set of all starting numbers that reach 1,000,000 or greater in four rounds or less. Therefore the smallest starting whole number is 62,496. To validate, I checked that starting with 62,496 results in 1,000,011 after four rounds, whereas 62,495 results in 999,995 after four rounds.

Lesson Summary

The formula, , describes the th term of the “double and add 5” game in terms of the starting number and . Use this formula to find the smallest starting whole number for the “double and add 5 game” that produces a result of 10,000,000 or greater in 15 rounds or less.

Exit Ticket (5 minutes)

Use the exit ticket to have students report their findings (the report step of the modeling cycle).

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Exit Ticket

Write a *brief* report about the answers you found to the *Double and Add 5* game problems. Include justifications for why your starting numbers are correct.

**MP.3**

Exit Ticket Sample Responses

Write a *brief* report about the answers you found to the *Double and Add 5* game problems. Include justifications for why your starting numbers are correct.

Results for finding the smallest starting number in the Double and Add 5 game:

1. Reaching 100 in three rounds or less: The starting number 9 results in 107 in round three. The starting number 8 results in 99 in round three, requiring another round to reach 100. Numbers 1–8 take more than three rounds to reach 100.
2. Reaching 1000 in three rounds or less: The starting number 121 results in 1003 in round three. The starting number 120 results in 995 in round three, requiring another round to reach 1000. All other whole numbers less than 120 take more than three rounds to reach 1000.
3. Reaching 1,000,000 in four rounds or less: The starting number 62,496 results in 1,000,011 in round four. The starting number 62,495 results in 999,995 in round four, requiring another round to reach 1,000,000. All other whole numbers less than 62,594 take more than three rounds to reach 1,000,000.

Problem Set Sample Solutions

1. Your older sibling came home from college for the weekend and showed you the following sequences (from her homework) that she claimed were generated from initial values and recurrence relations. For each sequence, find an initial value and recurrence relation that describes the sequence. (Your sister showed you an answer to the first problem.)
   1. (0, 2, 4, 6, 8, 10, 12, 14, 16, …)

and for

* 1. (1, 3, 5, 7, 9, 11, 13, 15, 17, …)

and for

* 1. (14, 16, 18, 20, 22, 24, 26, …)

and for

* 1. (14, 21, 28, 35, 42, 49, …)

and for

* 1. (14, 7, 0, –7, –14, –21, –28, –35, …)

and for

* 1. (2, 4, 8, 16, 32, 64, 128, …)

and for

* 1. (3, 6, 12, 24, 48, 96, …)

and for

* 1. (1, 3, 9, 27, 81, 243, …)

and for

* 1. (9, 27, 81, 243, …)

and for

1. Answer the following questions about the recursive sequence generated by initial value, , and recurrence relation, for .
   1. Find a formula for in terms of powers of 4.

* 1. Your friend, Carl, says that he can describe the th term of the sequence using the formula, Is Carl correct? Write one or two sentences using the recurrence relation to explain why or why not.

Yes. The recurrence relation, for , means that the next term in the sequence is always 4 times larger than the current term, i.e., one more power of 4. Therefore, the nth term will be n powers of 4, or .

1. The expression, , describes the th term of the *Double and Add 5* game in terms of the starting number and . Verify that it does describe the th term by filling out the tables below for parts (b) through (e). (The first table is done for you.)
   1. Table for .

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* 1. Table for .

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1. Bilbo Baggins stated to Samwise Gamgee, “Today, Sam, I will give you $1. Every day thereafter for the next 14 days, I will take the previous day’s amount, double it and add $5 and give that new amount to you for that day.”
   1. How much did Bilbo give Sam on day 15? (Hint: You don’t have to compute each term.)

. Bilbo gave Sam $196,603 on day 15.

* 1. Did Bilbo give Sam more than $350,000 altogether?

Yes. He gave $98,299 on day 14, $49,147, on day 13, $24,571 on day 12, and so on.

1. The formula, , describes the th term of the *Double and Add 5* game in terms of the starting number and . Use this formula to find the smallest starting whole number for the *Double and Add 5* game that produces a result of 10,000,000 or greater in 15 rounds or less.

Solving for results in .

Hence, 301 is the smallest starting whole number that will reach 10,000,000 in 15 rounds or less.